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Note on the representation of angular momentum by complex differential forms

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Abstract. The standard representation of angular momentum on Bargmann's Hilbert space of analytic functions is extended such that domains of operators include non-analytic functions, which are square integrable. Generalized eigenfunctions of angular momentum are proposed, which contain arbitrary complex constants. The eigenfunctions are shown to satisfy a basic requirement which follows from group theory.

Already in early treatises on group theory and quantum mechanics [1], one has encountered the suggestion of representing operators for angular momentum on the basis of two complex variables ξ and η. The idea can be developed in a mathematically sound manner [2] by situating oneself in a Hilbert space of analytic functions [3]. The ensuing representation of angular momentum forms part of common knowledge [4], and has been employed in various problems. Examples are given in [2, 5–8]. One can establish a correspondence with the boson calculus for angular momentum [9] via the identity

$$\xi^m \eta^n = \langle 0| a^m b^n \exp(\xi a^\dagger) \exp(\eta b^\dagger) |0\rangle \tag{1}$$

where m, n are non-negative integers, and $|0\rangle$ denotes a vacuum state. The ladder operators $a, a^\dagger$ and $b, b^\dagger$ satisfy boson commutation relations. The commutators $[a, b]$ and $[a, b^\dagger]$ equal zero.

For some applications, non-analytic functions cannot be excluded from the domain of the operators for angular momentum [10]. The same is true if one wishes to investigate certain group-theoretical properties of the eigenfunctions for angular momentum [11]. In these cases, the representation must be constructed anew, allowing now for forms that depend on the four independent complex variables $\xi, \eta, \bar{\xi}, \bar{\eta}$. This is precisely what we shall do in the present note. As a Hilbert space we shall choose $L^2(\mathbb{C}^2)$, the space of square-integrable functions on $\mathbb{C}^2$. The extended representation of angular momentum will give rise to a discussion of several new points.

Within the framework of non-relativistic quantum mechanics, the eigenvalue problem for angular momentum can be treated as follows [12]: one assumes the existence of three self-adjoint operators $\{J_x, J_y, J_z\}$ which act on a Hilbert space H, and satisfy the commutation relations

$$[J_k, J_l] = \mathrm{i} \sum_m \epsilon_{klm} J_m \tag{2}$$

where ϵ_{klm} denotes the Levi-Civita symbol. Subsequently, one demonstrates [13] that operators J_k and $J^2 \equiv \sum_l J_l^2$ commute, and possess a countable number of simultaneous eigenstates $\{|jm\rangle\}$, with

$$J_z|jm\rangle = m|jm\rangle \qquad J^2|jm\rangle = j(j+1)|jm\rangle \tag{3}$$

and

$$\langle j'm'|jm\rangle = \delta_{j'j}\delta_{m'm}. \tag{4}$$

The usual choice $k = z$ has been made. Quantum number j takes on the values $0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$, and for each j, quantum number m runs from $-j$ to $+j$ with unit steps. Finally, one demonstrates that the linear combinations

$$J_+ = J_x + \mathrm{i}J_y \qquad J_- = J_x - \mathrm{i}J_y \tag{5}$$

act as ladder operators according to

$$J_\pm|jm\rangle = [(j \mp m)(j \pm m + 1)]^{1/2}|jm \pm 1\rangle. \tag{6}$$

Together with (3), the above result implies that states $\{|jm\rangle : j \text{ fixed}\}$ span a $(2j+1)$-dimensional subspace of $\mathbf{H}$, which is invariant under the action of operators $\{J_x, J_y, J_z\}$.

As a realization of the abstract space $\mathbf{H}$, we choose a Hilbert space of functions $\mathcal{H}$. We seek to represent each component J_k of the angular-momentum operator by a complex differential form $\mathcal{J}_k$ that acts on $\mathcal{H}$. The new operators will be obtained by applying Stone's theorem [14] to a one-parameter group defined on $\mathcal{H}$. In constructing the latter, we set quantum number j equal to its lowest non-trivial value, given by $j = \frac{1}{2}$. One then has $\langle \frac{1}{2}m'|J_k|\frac{1}{2}m\rangle = \frac{1}{2}\sigma_k$, where $\{\sigma_x, \sigma_y, \sigma_z\}$ are the Pauli matrices. These are generators of SU(2), in view of the identity

$$U(\hat{e}, \phi) \equiv \exp(-\tfrac{1}{2}\mathrm{i}\phi\hat{e}\cdot\boldsymbol{\sigma}) = \cos(\tfrac{1}{2}\phi)\mathbf{1} - \mathrm{i}\sin(\tfrac{1}{2}\phi)\hat{e}\cdot\boldsymbol{\sigma} \tag{7}$$

where ϕ denotes a real parameter, and $\hat{e}$ a unit vector in $\mathbb{R}^3$.

Matrices $\{U(\hat{e}, \phi)\}$ make up a unitary one-parameter group on $\mathbb{C}^2$, with generator $\frac{1}{2}\hat{e}\cdot\boldsymbol{\sigma}$. We are thus led to consider complex-valued functions on $\mathbb{C}^2$, and introduce the operator

$$P_U f(\xi, \eta, \bar{\xi}, \bar{\eta}) = f(\xi', \eta', \bar{\xi}', \bar{\eta}') \qquad (\xi', \eta') = (\xi, \eta)U. \tag{8}$$

This definition complies with literature conventions†. Note that a complex number contains two degrees of freedom, so ξ and its complex conjugate $\bar{\xi}$ may be treated as independent variables. Space $\mathcal{H}$ is taken to be the Hilbert space $L^2(\mathbb{C}^2)$ of square-integrable functions on $\mathbb{C}^2$, so the scalar product between two functions is given by

$$\langle f, g\rangle = \int\int \mathrm{d}^2\xi\, \mathrm{d}^2\eta\, \bar{f}(\xi, \eta, \bar{\xi}, \bar{\eta})g(\xi, \eta, \bar{\xi}, \bar{\eta}). \tag{9}$$

The integrals must be evaluated on the basis of the prescription $\int \mathrm{d}^2\xi = \int\int \mathrm{d}x\,\mathrm{d}y$, with $\xi = x + \mathrm{i}y$.

From definitions (7) and (8) we infer that the set $\{P_U : \phi \in \mathbb{R}\}$ forms a one-parameter group on $L^2(\mathbb{C}^2)$. The set $\mathcal{S}$ of functions of rapid decrease on $\mathbb{C}^2$ is dense in $L^2(\mathbb{C}^2)$. By employing the dominated-convergence theorem‡, one proves that for each $f \in \mathcal{S}$ the function $P_U f$ converges to f in the L^2 norm as parameter ϕ tends to zero. Therefore, the one-parameter group $\{P_U\}$ is strongly continuous. It is also unitary, because integral (9) is invariant under any special unitary transformation of vector (ξ, η). We recall that vector $(\mathrm{Re}\,\xi, \mathrm{Im}\,\xi, \mathrm{Re}\,\eta, \mathrm{Im}\,\eta)^{\mathrm{T}}$ transforms with SO(4), if vector $(\xi, \eta)^{\mathrm{T}}$ transforms with SU(2).

† See [11, p 106].
‡ See for instance [14, section I.3].

We are now in a position to apply Stone's theorem, so that we may write

$$P_U f(\xi,\eta,\bar{\xi},\bar{\eta}) = \exp(-\mathrm{i}\phi K) f(\xi,\eta,\bar{\xi},\bar{\eta}). \tag{10}$$

For each $f \in \mathcal{S}$, the form $\mathrm{i}\phi^{-1}(P_U f - f)$ converges to Kf in the L^2 norm as ϕ goes to zero. Employing (8) as well as the dominated-convergence theorem, we find for the generator

$$Kf = \hat{e} \cdot \mathcal{J} f \tag{11}$$

with

$$\begin{aligned}
\mathcal{J}_x &= \frac{1}{2}\left[\eta\frac{\partial}{\partial\xi} + \xi\frac{\partial}{\partial\eta} - \bar{\eta}\frac{\partial}{\partial\bar{\xi}} - \bar{\xi}\frac{\partial}{\partial\bar{\eta}}\right] \\
\mathcal{J}_y &= \frac{\mathrm{i}}{2}\left[\eta\frac{\partial}{\partial\xi} - \xi\frac{\partial}{\partial\eta} + \bar{\eta}\frac{\partial}{\partial\bar{\xi}} - \bar{\xi}\frac{\partial}{\partial\bar{\eta}}\right] \\
\mathcal{J}_z &= \frac{1}{2}\left[\xi\frac{\partial}{\partial\xi} - \eta\frac{\partial}{\partial\eta} - \bar{\xi}\frac{\partial}{\partial\bar{\xi}} + \bar{\eta}\frac{\partial}{\partial\bar{\eta}}\right].
\end{aligned} \tag{12}$$

In terms of real variables x and y, with $\xi = x + \mathrm{i}y$, the partial derivatives with respect to ξ and $\bar{\xi}$ can be expressed as

$$\frac{\partial}{\partial\xi} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right) \qquad \frac{\partial}{\partial\bar{\xi}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right). \tag{13}$$

Now relations $\partial\xi/\partial\xi = 1$ and $\partial\xi/\partial\bar{\xi} = 0$ can be directly verified. Moreover, the self-adjointness of operator K can be made explicit, because each $\mathcal{J}_k$ is seen to be a linear combination with real coefficients of operators of the form $\mathrm{i}u\partial/\partial v$, where u and v stand for any two variables of the set $\{\mathrm{Re}\,\xi, \mathrm{Im}\,\xi, \mathrm{Re}\,\eta, \mathrm{Im}\,\eta\}$†.

By construction, operators (12) obey the same commutation relations as the Pauli matrices. Therefore, we have found a representation of angular momentum in terms of operators that are self-adjoint on the Hilbert space $L^2(\mathbb{C}^2)$. In literature [7, 8] representations have been proposed which are similar to (12). To be specific, the choice $z_{11} = \xi, z_{12} = \eta, z_{21} = -\bar{\eta}, z_{22} = \bar{\xi}$ in [7, section 5] also provides us with operators (12). The same goes‡ for the choice $\xi_1 = \zeta_1 = \xi$ and $\xi_2 = \zeta_2 = \eta$ in equation (33) of [8]. Upon discarding partial derivatives with respect to $\bar{\xi}$ and $\bar{\eta}$, operators (12) assume their standard form [2, 4–6]. However, rather than on $L^2(\mathbb{C}^2)$, they are now self-adjoint on the Hilbert space of analytic functions [2]. For some physical applications this may be a drawback [10]. Below we list some advantages and new features of the extended representation.

(1) One checks that operators (12) are invariant under the transformation

$$\xi' = a\xi + b\bar{\eta} \qquad \eta' = -b\bar{\xi} + a\eta \tag{14}$$

where a and b denote arbitrary complex numbers. In case the sum $|a|^2 + |b|^2$ equals unity, (14) corresponds to a special unitary transformation of vector $(\xi, \bar{\eta})^{\mathrm{T}}$. The standard counterpart of (12) is not invariant under transformation (14), unless b is chosen to be zero.

(2) We set out to find elements $\{v_{jm}\}$ of $\mathcal{S}$ which represent the abstract states $\{|jm\rangle\}$. To that end, we need to represent ladder operators J_+ and J_- on the basis of (12). They come out as

$$\mathcal{J}_+ = \xi\frac{\partial}{\partial\eta} - \bar{\eta}\frac{\partial}{\partial\bar{\xi}} \qquad \mathcal{J}_- = \eta\frac{\partial}{\partial\xi} - \bar{\xi}\frac{\partial}{\partial\bar{\eta}}. \tag{15}$$

† The self-adjointness of these operators is discussed in [14, section VIII.5].

‡ One should still adapt a few coefficients. Note that relations (33) and (34) of [8] are not consistent with each other.

From (6) conditions $J_+|jj\rangle = J_-|j-j\rangle = 0$ are found, which can be satisfied by letting $|jj\rangle \to p_1(\xi,\bar\eta)$ and $|j-j\rangle \to p_2(\bar\xi,\eta)$.

For functions p_1 and p_2 we choose polynomials. States $J_+^{2j}|j-j\rangle$ and $|jj\rangle$, as well as states $J_-^{2j}|jj\rangle$ and $|j-j\rangle$ are linearly dependent, so powers higher than $2j$ may not occur. In view of eigenvalue equations (3), p_1 must be a homogeneous polynomial of order $2j$. We act $j-m$ times with operator $\mathcal{J}_-$ on polynomial $p_1 = \sum_{k=0}^{2j} a_k \xi^{2j-k}\bar\eta^k$, with $\{a_k\}$ complex numbers. After repeatedly using (6), we see that functions $\{v_{jm}\}$ can be cast into the following form

$$v_{jm} = \frac{N_j \exp[-\gamma_j(|\xi|^2+|\eta|^2)]}{[(j+m)!(j-m)!]^{1/2}} \sum_{k=0}^{j+m}\sum_{l=0}^{j-m} c_{k+l}^{(j)}(-1)^k \binom{j+m}{k}\binom{j-m}{l} \xi^{j+m-k}\eta^{j-m-l}\bar\xi^l\bar\eta^k \tag{16}$$

where N_j and γ_j are real and positive constants. The exponential factor has been added so as to ensure that functions $\{v_{jm}\}$ belong to $\mathcal{S}$. Its presence does not affect the validity of equations (3) and (6), because of the identity

$$\mathcal{J}_k(|\xi|^2+|\eta|^2) = 0 \tag{17}$$

which is a consequence of (8), (10), and the unitarity of matrix U.

The complex numbers $\{c_n^{(j)}\}$ are not subject to any conditions. Hence, the extended representation (12) leaves us a lot of freedom in translating equations (3) and (6) into functional language. Upon choosing $c_n^{(j)} = \delta_{n,0}$ in (16), we recover the well known [2, 4–6] monomials that are associated to the standard counterpart of (12).

We still have to examine the orthonormality of the set $\{v_{jm}\}$, and compute the normalization factors $\{N_j\}$. The scalar product (9) of v_{jm} and $v_{j'm'}$ can be elaborated on the basis of the identity

$$\int \mathrm{d}^2\xi\,\bar\xi^m\xi^n \exp(-|\xi|^2) = \pi n!\delta_{mn} \tag{18}$$

where n, m denote non-negative integers. The two ensuing Kronecker symbols reduce the number of summations to three, and make the scalar product identically zero in the case $m \neq m'$. This brings us to

$$\langle v_{j'm'}, v_{jm}\rangle = \delta_{m'm}\frac{\pi^2(-1)^{j'+j}N_{j'}N_j}{(\gamma_{j'}+\gamma_j)^{j'+j+2}}\left[\frac{(j'+m)!(j'-m)!}{(j+m)!(j-m)!}\right]^{1/2}$$
$$\times\sum_{k=0}^{j+m}\sum_{l=0}^{j-m}(-1)^l\bar c_{j'-j+k+l}^{(j')}c_{k+l}^{(j)}\binom{j+m}{k}\binom{j-m}{l}F(j'jm;kl). \tag{19}$$

We have defined a factor

$$F(j'jm;kl) = \sum_{n=0}^{j'-m}(-1)^n\frac{(j+m+n-k)!(j'-m+k-n)!}{n!(j+m+n-k-l)!(j'-m-n)!(j'-j+k+l-n)!} \tag{20}$$

and assumed the inequality $j' \geqslant j$, without loss of generality.

The above result calls for the use of the following well known [15] formulae

$$\sum_{n=0}^{p}(-1)^{n+p}\binom{p}{n}n^q = p!\delta_{pq} \qquad \text{with } q \leqslant p \tag{21}$$

and p, q non-negative integers. In evaluating (20), the cases $j+m \leqslant k+l \leqslant 2j$, $j-m \leqslant k+l \leqslant j+m$, and $0 \leqslant k+l \leqslant j-m$ must be handled separately. In the third case F can be written as a sum of contributions of the form (21), with $p = j'-j+k+l$

and $0 \leqslant q \leqslant k + l$. It follows that $F(j'jm; kl)$ vanishes for $j' > j$, and equals $(-1)^l$ for $j' = j$. The two other cases yield the same result. The r.h.s. of (19) can now be computed with the help of the addition theorem

$$\sum_{k=0}^{p} \binom{m}{k} \binom{n}{p-k} = \binom{m+n}{p}. \tag{22}$$

We end up with

$$\langle v_{j'm'}, v_{jm} \rangle = \delta_{j'j} \delta_{m'm} \frac{\pi^2 N_j^2}{(2\gamma_j)^{2j+2}} \sum_{n=0}^{2j} |c_n^{(j)}|^2 \binom{2j}{n}. \tag{23}$$

A comparison with (4) gives the normalization factors $\{N_j\}$.

It should be emphasized that the orthonormal set $\{v_{jm}\}$ spans a closed space, which is a true subspace of $L^2(\mathbb{C}^2)$. Indeed, with the help of identities similar to (18) and (21), one demonstrates that the function of rapid decrease

$$\left(\frac{2}{1 + \gamma_0} - |\xi|^2 - |\eta|^2 \right) \exp[-|\xi|^2 - |\eta|^2] \tag{24}$$

is orthogonal to the set $\{v_{jm}\}$. The situation is the same as for the Schrödinger representation of angular momentum, where the spherical harmonics $\{Y_{lm}(\hat{\boldsymbol{r}})\}$ represent states $\{|jm\rangle\}$ for $j = l = 0, 1, 2, \ldots$. Upon multiplying by a suitably chosen radial function, the spherical harmonics are transformed into an orthonormal set on the Hilbert space $L^2(\mathbb{R}^3)$. This set is, however, incomplete.

(3) Under the action of operators $\{P_U\}$ the functions $\{v_{jm} : j \text{ fixed}\}$ transform amongst each other. Therefore, they form the basis of a $(2j + 1)$-dimensional unitary representation of SU(2). It is irreducible, because for each function w that belongs to the jth eigenspace of operator $\mathcal{J}^2$, the set $\{P_U w : U \in \mathrm{SU(2)}\}$ contains $2j + 1$ linearly independent elements. One can now prove [11] that the sum

$$s_j(\xi, \eta, \bar{\xi}, \bar{\eta}) = \sum_{m=-j}^{+j} |v_{jm}(\xi, \eta, \bar{\xi}, \bar{\eta})|^2 \tag{25}$$

is invariant under operators $\{P_U\}$.

To confirm the foregoing statement, we evaluate function s_j by substituting (16). After making a few changes of summation index, identities (21) can be used again, with the outcome

$$s_j(\xi, \eta, \bar{\xi}, \bar{\eta}) = \frac{(2\gamma_j)^{2j+2}}{\pi^2 (2j)!} (|\xi|^2 + |\eta|^2)^{2j} \exp[-2\gamma_j(|\xi|^2 + |\eta|^2)]. \tag{26}$$

The normalization constant N_j has been eliminated. On account of (17), one now arrives at the satisfactory result

$$\mathcal{J}_k s_j(\xi, \eta, \bar{\xi}, \bar{\eta}) = 0 \tag{27}$$

for $k = x, y, z$. Notice that only within the framework of the extended representation (12), the invariance of function s_j can be discussed. The latter is not analytic, and, hence, does not belong to Bargmann's Hilbert space.

In conclusion, the eigenvalue problem for angular momentum has been formulated and investigated on the Hilbert space $L^2(\mathbb{C}^2)$. For arbitrary quantum number j, we have proposed a set of eigenfunctions $\{v_{jm}\}$ that contains $2j + 1$ free complex constants. As an application, one might analyse the role of these in the derivation of special-function identities via coupling of angular momentum [16]. We have verified a group-theoretical statement which states that the sum $\sum_{m=-j}^{+j} |v_{jm}|^2$ is invariant under SU(2) operations.

Finally, we point out that one can also choose $L^2(\mathbb{C}^3)$ as a Hilbert space, and thus establish a representation in terms of the complex variables ξ, η, ζ, and their conjugates. Starting from matrices $\langle 1m'|J_k|1m\rangle$, with $k = x, y, z$, one obtains

$$\begin{aligned} \mathcal{J}_x &= \frac{1}{\sqrt{2}}\left[\eta\frac{\partial}{\partial\xi} + (\xi+\zeta)\frac{\partial}{\partial\eta} + \eta\frac{\partial}{\partial\zeta} - \bar{\eta}\frac{\partial}{\partial\bar{\xi}} - (\bar{\xi}+\bar{\zeta})\frac{\partial}{\partial\bar{\eta}} - \bar{\eta}\frac{\partial}{\partial\bar{\zeta}}\right] \\ \mathcal{J}_y &= \frac{\mathrm{i}}{\sqrt{2}}\left[\eta\frac{\partial}{\partial\xi} - (\xi-\zeta)\frac{\partial}{\partial\eta} - \eta\frac{\partial}{\partial\zeta} + \bar{\eta}\frac{\partial}{\partial\bar{\xi}} - (\bar{\xi}-\bar{\zeta})\frac{\partial}{\partial\bar{\eta}} - \bar{\eta}\frac{\partial}{\partial\bar{\zeta}}\right] \\ \mathcal{J}_z &= \left[\xi\frac{\partial}{\partial\xi} - \zeta\frac{\partial}{\partial\zeta} - \bar{\xi}\frac{\partial}{\partial\bar{\xi}} + \bar{\zeta}\frac{\partial}{\partial\bar{\zeta}}\right]. \end{aligned} \tag{28}$$

The set $\mathcal{S}$ may serve as a domain of the above operators.

It is interesting to remark that the invariant sums $\{s_j\}$ are now of a more complicated structure than before. For $j = 1$, the three eigenfunctions of $\mathcal{J}_z$ and $\mathcal{J}^2$ can be taken as ξp, ηp, and ζp, where factor $p = \pi^{-3/2}\exp[-\frac{1}{2}(|\xi|^2+|\eta|^2+|\zeta|^2)]$ ensures normalization. Both sum s_1 and factor p contain the invariant $|\xi|^2 + |\eta|^2 + |\zeta|^2$. Since the five eigenfunctions for $j = 2$ can be taken as $2^{-1/2}\xi^2 p$, $\xi\eta p$, $3^{-1/2}(\eta^2 + \xi\zeta)p$, $\eta\zeta p$, and $2^{-1/2}\zeta^2 p$, sum s_2 contains a new invariant, defined by the relation

$$\mathcal{J}_k[\tfrac{1}{2}|\xi|^4 + |\xi|^2|\eta|^2 + \tfrac{1}{3}|\eta^2 + \xi\zeta|^2 + |\eta|^2|\zeta|^2 + \tfrac{1}{2}|\zeta|^4] = 0 \tag{29}$$

with $k = x, y, z$. This example illustrates that by studying representations of angular momentum on the space $L^2(\mathbb{C}^{2j+1})$, one might discover new and useful results.

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